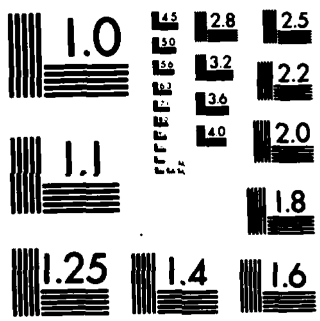
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GLOBAL EXISTENCE AND ASYMPTOTICS IN ONE-DIMENSIONAL NONLINEAR VISCOELASTICITY

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ABSTRACT

In this paper we survey recent results concerning global existence and decay of smooth solutions of certain quasilinear hyperbolic Volterra equations which provide models for the motion of one-dimensional viscoelastic solid of the Boltzmann type. We also sketch the derivation of these equations from physical principles, discuss the physically appropriate assumptions, and prove a special case of a new existence theorem for the Cauchy problem.

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GLOBAL EXISTENCE AND ASYMPTOTICS IN
ONE-DIMENSIONAL NONLINEAR VISCOELASTICITY

W. J. Hrusa[1,2] and J. A. Nohel[1]

1. Introduction

For nonlinear elastic bodies, the balance laws of continuum mechanics lead to equations of motion (of hyperbolic type) which have the property that smooth solutions may break down in finite time due to the formation of shock waves. Some material models of physical interest incorporate a nonlinear "elastic-type" response in conjunction with a natural dissipative mechanism. For such materials it is important to understand the effects of dissipation on solutions of the equations of motion.

Some dissipative mechanisms (e.g., viscosity of the rate type in one space dimension) are so powerful that globally defined smooth solutions exist, even for very large initial data. A much more subtle type of dissipation, due to memory effects, arises in viscoelasticity of the Boltzmann type.

In this paper we discuss global existence and decay of smooth solutions of certain quasilinear hyperbolic Volterra equations which provide models for the motion of nonlinear viscoelastic solids of the Boltzmann type. In Section 2 we formulate the dynamic problems to be considered and discuss the relevant assumptions. In Section 3 we give a survey of known results. Theorem 3.1 is new; the complete proof will appear elsewhere [17]. Finally, in Section 4, we prove a special case of Theorem 3.1.

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We restrict our attention throughout to one-dimensional motions. Although the details have not been carried out completely, analogous results can be obtained for multidimensional viscoelastic solids of the Boltzmann type. This is discussed briefly in [7]. Local existence results which are applicable to multidimensional bodies occupying all of space have been given by Grimmer and Zeman [12].

We close the introduction with some remarks on notation. Let D be a subset of $\mathbb{R} \times \mathbb{R}$. For a function $w : D \to \mathbb{R}$ we use subscripts x and t (or τ) to indicate partial differentiation with respect to the first and second argument, respectively. Moreover, we use the same symbol w to denote the mapping $t \mapsto w(\cdot,t)$ when there is no danger of confusion. A prime is used to denote the derivative of a function of a single variable, and the symbol $:=$ indicates an equality in which the left hand side is defined by the right hand side. All derivatives should be interpreted in the sense of distributions.

2. Formulation of Dynamic Problems

Consider the longitudinal motion of a homogeneous one-dimensional body that occupies the interval B in a reference configuration (which we assume to be a natural state) and has unit reference density. We denote by $u(x,t)$ the displacement at time t of the particle with reference position x (i.e., $x + u(x,t)$ is the position at time t of the particle with reference position x), in which case the strain is given by

$$\varepsilon(x,t) := u_x(x,t) \quad . \tag{2.1}$$

For smooth displacements, the equation of balance of linear momentum here takes the form

$$u_{tt}(x,t) = \sigma_x(x,t) + f(x,t), \quad x \in B, \ t \geq 0 \quad , \tag{2.2}$$

where σ is the stress and f is the (known) body force. Equation (2.2) must be supplemented with a constitutive assumption (stress-strain relation) which characterizes the type of material composing the body.

If the body is elastic, then the stress depends on the strain through a constitutive equation of the form

$$\sigma(x,t) = \phi(\varepsilon(x,t)) \quad , \tag{2.3}$$

where ϕ is an assigned smooth function with $\phi(0) = 0$, and the resulting equation of motion is

$$u_{tt} = \phi(u_x)_x + f \quad . \tag{2.4}$$

Experience indicates that stress increases with strain, at least near equilibrium, so it is natural to assume that $\phi'(0) > 0$. Lax [18] and MacCamy and Mizel [23] have shown that (2.4) (with $f \equiv 0$) does not generally have globally defined smooth solutions no matter how smooth (and small) the initial data are.

For viscoelastic materials of the rate type, the stress depends on the strain rate as well as the strain. A simple model corresponds to the constitutive relation

$$\sigma(x,t) = \phi(\varepsilon(x,t)) + \lambda\varepsilon_t(x,t) \quad , \tag{2.5}$$

where ϕ is as above and λ is a positive constant, which leads to the equation

$$u_{tt} = \phi(u_x)_x + \lambda u_{xtx} + f \quad . \tag{2.6}$$

Greenberg, MacCamy, and Mizel [11] have shown that the Dirichlet initial-boundary value problem for (2.6) has a unique globally defined smooth solution provided that the initial data are sufficiently smooth. Viscosity of the rate type is so powerful that global smooth solutions exist even if the initial data are very large. Similar results for more general viscoelastic materials of the rate type have been obtained by Dafermos [3] and MacCamy [19].

Experience indicates that in certain materials, the stress at a material point x depends on the entire temporal history of the strain at x. In 1876, Boltzmann [1] proposed the constitut... ...on

$$\sigma(x,t) = c\varepsilon(x,t) - \int_0^\infty m(s)\varepsilon(x,t-s)ds \quad , \tag{2.7}$$

where c is a positive constant and m is positive, decreasing, integrable, and satisfies

$$c - \int_0^\infty m(s)ds > 0 \quad . \tag{2.8}$$

The history of the strain up to time $t = 0$ is assumed to be known.

The constant c measures the instantaneous response of stress to strain, and the first two conditions on m say that the stress "relaxes" as time increases and that deformations which occurred in the distant past have less influence on the present stress than those which occurred in the recent past. Equation (2.8) also has an important mechanistic interpretation. In statics, i.e. $\sigma(x,t) \equiv \bar{\sigma}(x)$ and $\varepsilon(x,t) \equiv \bar{\varepsilon}(x)$, equation (2.7) reduces to

$$\bar{\sigma}(x) = (c - \int_0^\infty m(s)ds)\bar{\varepsilon}(x) \quad , \tag{2.9}$$

and thus (2.8) states that the equilibrium stress modulus is positive.

A natural nonlinear generalization of (2.7) is provided by the constitutive equation

$$\sigma(x,t) = \phi(\varepsilon(x,t)) - \int_0^\infty m(s)\psi(\varepsilon(x,t-s))ds \tag{2.10}$$

where ϕ and ψ are assigned smooth functions with

$$\phi(0) = \psi(0) = 0, \ \phi'(0) > 0, \ \psi'(0) > 0 \quad , \tag{2.11}$$

and m is positive, decreasing, integrable, and satisfies

$$\phi'(0) - (\int_0^\infty m(s)ds)\psi'(0) > 0 \quad . \tag{2.12}$$

It is convenient to define the relaxation function a by

$$a(t) := \int_t^\infty m(s)ds, \quad t \in [0,\infty) \quad , \tag{2.13}$$

and the equilibrium stress function χ by

$$\chi(\xi) := \phi(\xi) - a(0)\psi(\xi), \quad \xi \in \mathbb{R} \quad . \tag{2.14}$$

If m satisfies the preceding conditions then a is positive, decreasing, and convex, and $\chi'(0) > 0$.

We note that $a' \equiv -m$. Thus (2.10) can be written in the form

$$\sigma(x,t) = \phi(\varepsilon(x,t)) + \int_0^\infty a'(s)\phi(\varepsilon(x,t-s))ds \tag{2.15}$$

and (letting $\tau := t-s$) also in the form

$$\sigma(x,t) = \phi(\varepsilon(x,t)) + \int_{-\infty}^t a'(t-\tau)\phi(\varepsilon(x,\tau))d\tau \quad . \tag{2.16}$$

The corresponding equation of motion is

$$u_{tt}(x,t) = \phi(u_x(x,t))_x + \int_{-\infty}^t a'(t-\tau)\phi(u_x(x,\tau))_x d\tau \tag{2.17}$$
$$+ f(x,t), \; x \in B, \; t > 0 \quad .$$

Observe that a', rather than a, appears in equations (2.15), (2.16), and (2.17). In this paper, we are normalizing a so that $a(t) \to 0$ as $t \to \infty$. (See (2.13).) The reader is cautioned that other normalizations are frequently used.

An appropriate dynamic problem is to determine a smooth function $u : B \times (-\infty,\infty) \to \mathbb{R}$ which satisfies equation (2.17) for $t > 0$, together with suitable boundary conditions if B is bounded, and

$$u(x,t) = v(x,t) \qquad x \in B, \; t < 0 \quad , \tag{2.18}$$

where v is a given smooth function. The history value problem (2.17), (2.18) can be reduced to an initial value problem as follows. Define a new forcing function g by

$$g(x,t) := f(x,t) + \int_{-\infty}^0 a'(t-\tau)\phi(v_x(x,\tau))_x d\tau, \tag{2.19}$$
$$x \in B, \; t < 0 \quad ,$$

and initial data u_0, u_1 by

$$u_0(x) = v(x,0), \; u_1(x) = v_t(x,0), \; x \in B \quad . \tag{2.20}$$

It is clear that u is a solution of (2.17), (2.18) if and only if it is a solution of the initial value problem

$$u_{tt}(x,t) = \phi(u_x(x,t))_x + \int_0^t a'(t-\tau)\phi(u_x(x,\tau))_x d\tau \qquad (2.21)$$

$$+ g(x,t),\ x \in B,\ t > 0\ ,$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in B\ . \qquad (2.22)$$

Conversely, the initial value problem (2.21), (2.22) can be converted to a history value problem of the form (2.17), (2.18) by constructing suitable functions v and f. (Of course, such a procedure does not uniquely determine a history value problem.) For consistency, we state all results for initial value problems. Clearly, there are analogous statements for history value problems.

We consider pure initial value problems (Cauchy problems) with $B = \mathbb{R}$, as well as initial-boundary value problems with $B = [0,1]$ and boundary conditions of Dirichlet, Neumann, or mixed type, i.e.

$$u(0,t) = u(1,t) = 0, \quad t > 0\ , \qquad (2.23)$$

$$u_x(0,t) = u_x(1,t) = 0, \quad t > 0\ , \qquad (2.24)$$

or

$$u(0,t) = u_x(1,t) = 0, \quad t > 0\ . \qquad (2.25)$$

The physical interpretation of (2.23) is clear. Under certain appropriate conditions, (2.24) is equivalent to

$$\sigma(0,t) = \sigma(1,t) = 0,\ t > 0\ . \qquad (2.26)$$

See, for example, [7]. (A similar comment applies to (2.25).)

For initial-boundary value problems, the initial data and g should be compatible with the boundary conditions. For example, suppose that u is a classical solution of (2.21), (2.22), (2.23), (with $g \equiv 0$ for simplicity) on $[0,1] \times [0,T]$ for some $T > 0$. Differentiating (2.23) twice with respect to t yields

$$u_t(0,t) = u_t(1,t) = u_{tt}(0,t) = u_{tt}(1,t) = 0 \qquad (2.27)$$

$$\forall\ t \in [0,T]\ .$$

If (2.21), (2.22), (2.23), and (2.27) are to hold at $t = 0$ (and $\phi'(u_0')$ does not vanish), then u_0 and u_1 must satisfy

$$u_0(0) = u_0(1) = u_1(0) = u_1(1) = u_0''(0) = u_0''(1) = 0 \quad . \tag{2.28}$$

Violation of the above condition should be interpreted as a singularity in the initial data on the boundary. Due to the hyperbolic nature of equation (2.21), such a singularity would try to propagate away from the boundary and into the interior. Analogous compatibility conditions are required for (2.24) and (2.25). (If $g \not\equiv 0$, then the compatibility conditions also involve g.)

3. Survey of Results

Observe that if a' vanishes identically, then (2.21) reduces to an undamped quasilinear wave equation. If $a' \not\equiv 0$ and the appropriate sign conditions are satisfied, the memory term in (2.21) induces a weak type of dissipation. A great deal of information concerning the strength of this dissipative mechanism is contained in the work of Coleman and Gurtin [2] on the growth and decay of acceleration waves in materials with memory. Roughly speaking, they showed that (under physically natural assumptions) the amplitude of a certain type of weak singularity (involving jump discontinuities in second derivatives of u) decays to zero as $t \to \infty$, provided its initial amplitude is sufficiently small. On the other hand, the amplitude of such a singularity may become infinite in finite time if its initial amplitude is too large.

This suggests that (2.21) should have globally defined smooth solutions for sufficiently smooth and small data, and that smooth solutions can develop singularities in finite time if the data are suitably large. (Here we use the term data to mean initial data and forcing function.) Results of this type have been obtained by a number of authors. At the present time, the situation

concerning existence of global solutions for small data is quite well understood; less is known about the formation of singularities. It should be noted that several important ideas used in the analysis of (2.21) were motivated by the work of Nishida [27] and Matsumura [26] on quasilinear wave equations with frictional damping.

Local existence of smooth solutions to (2.21) can be established by more or less routine procedures. (See, for example, [7].) The local arguments require only positivity of ϕ' and smoothness of ϕ, ψ, a, and the data. In particular, they are insensitive to the "sign" of the memory term and the size of the data. However, rather delicate a priori estimates are needed to show that local solutions can be continued globally. These estimates rely crucially on the memory term having the correct sign and the data being small.

For the special case $\psi \equiv \phi$, global existence theorems have been established by MacCamy [21], Dafermos and Nohel [6], and Staffans [30]. In order to simplify our discussion of these results, let us assume that $g \equiv 0$ and consider the problem

$$u_{tt}(x,t) = \phi(u_x(x,t))_x + \int_0^t a'(t-\tau)\phi(u_x(x,\tau))_x d\tau \quad , \qquad x \in B,\ t > 0 \quad , \tag{3.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in B \quad . \tag{3.2}$$

The main hypotheses on ϕ and a are

$$\phi \in C^3(\mathbb{R}),\ \phi(0) = 0,\ \phi'(0) > 0 \quad , \tag{3.3}$$

$$a,\ a',\ a'' \in L^1(0,\infty) \quad , \tag{3.4}$$

$$a \text{ is strongly positive definite} \quad , \tag{3.5}$$

$$a(0) < 1 \quad . \tag{3.6}$$

(Some additional technical assumptions on a are used in [21] and [6].) We refer the reader to [29] and [30] for properties of strongly positive definite kernels. We note, however, that twice continuously differentiable a which

satisfy

$$(-1)^k a^{(k)}(t) \geq 0 \quad \forall\, t > 0,\ k = 0,1,2;\ a' \not\equiv 0 \quad , \tag{3.7}$$

are automatically strongly positive definite. (Corollary 2.2 of [29].) Condition (3.6), together with $\phi'(0) > 0$, simply states that $\chi'(0) > 0$.

Remark 3.1: We note that a' rather than a appears in equation (3.1). Our normalizations of a (with $a(\infty) = 0$) is different from that used in [21], [6], and [30]. For this reason, the conditions on a above are in a slightly different form than in [21], [6], and [30].

The assumptions needed on u_0 and u_1 vary slightly depending on the type of boundary conditions. Roughly speaking it is required that

$$u_0',\ u_0'',\ u_0''',\ u_1,\ u_1',u_1'' \in L^2(B) \tag{3.8}$$

and that the $L^2(B)$ norms of the functions listed in (3.8) be sufficiently small. In addition, the data must be compatible with the boundary conditions if B *is bounded. It is not assumed that the* $L^2(B)$ norm of u_0 is small. However, for certain initial-boundary value problems, this is implied by the Poincaré inequality and smallness of the $L^2(B)$ norm of u_0'.

Under the above assumptions, the initial value problem (3.1), (3.2), with $B = \mathbb{R}$, has a unique solution $u \in C^2(\mathbb{R} \times [0,\infty))$ such that

$$u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in C([0,\infty); L^2(\mathbb{R})) \quad . \tag{3.9}$$

Moreover, as $t \to \infty$,

$$u_{tt},\ u_{tx},\ u_{xx} \to 0 \ \text{ in } \ L^2(\mathbb{R}) \quad , \tag{3.10}$$

$$u_t,\ u_x,\ u_{tt},\ u_{tx},\ u_{xx} \to 0 \ \text{ uniformly on } \ \mathbb{R} \quad . \tag{3.11}$$

Similar conclusions hold for initial-boundary value problems for (3.1) with $B = [0,1]$ and boundary conditions (2.23), (2.24), or (2.25). The precise decay statement depends on the boundary conditions. For (2.23) or (2.25) (i.e., Dirichlet or mixed conditions),

$$u, u_t, u_x, u_{tt}, u_{tx}, u_{xx} \to 0 \text{ uniformly on } [0,1] \tag{3.12}$$

as $t \to \infty$, while for (2.24) (Neumann conditions),

$$u_x, u_{tt}, u_{tx}, u_{xx} \to 0 \text{ uniformly on } [0,1] \tag{3.13}$$

as $t \to \infty$. The difference is due to the fact that nontrivial rigid motions are possible under (2.24), but not under (2.23) or (2.25). See [21], [6], and [30] for the proofs. (The boundary conditions (2.25) are not discussed explicitly, but the same proofs apply with only trivial modifications.)

Remark 3.2: If, under boundary conditions (2.24), it is assumed that the data have zero average spatially then the solution will have zero average spatially and (3.13) can be replaced by (3.12). A Neumann problem can always be reduced to one in which the data have zero average by superposition of a rigid motion. (See, for example, [7] or [16].)

Remark 3.3: The above results remain valid if a suitably smooth and small forcing function g (which behaves properly as $t \to \infty$) is included in (3.1). See [6], [21], and [30]. (See also Theorem 3.1 below for an indication of the type of assumptions required of g.)

On the other hand, Hattori [13] has shown that if $\phi'(\xi) > 0$ $\forall\, \xi \in \mathbb{R}$ and $\phi'' \not\equiv 0$, then there are smooth initial data (compatible with the boundary conditions) for which the initial-boundary value problem (3.1), (3.2), (2.23), with $B = [0,1]$, does not have a globally defined smooth solution. Such data must necessarily be large in view of the aforementioned existence results. The precise manner in which loss of regularity occurs is not discussed in [13]. Markowich and Renardy [25] have obtained numerical evidence which indicates the formation of shock fronts in smooth solutions of the initial value problem (3.1), (3.2) with $B = \mathbb{R}$ and suitably large initial data.

The following idea of MacCamy reveals that there is a close similarity between (3.1) and a wave equation with frictional damping. Observe that $\phi(u_x)_x$ can be expressed in terms of u_{tt} through an inverse linear Volterra operator. An integration by parts can then be used to transfer a time derivative from u_{tt} to the resolvent kernel associated with a'. This introduces a frictional damping term and renders the memory term a linear perturbation of lower order.

More precisely, the (scalar) linear Volterra operator L defined by

$$(Lw)(t) := w(t) + \int_0^t a'(t-\tau)w(\tau)d\tau, \ t \geq 0 \quad , \tag{3.14}$$

is invertible with inverse given by

$$(L^{-1}\bar{w})(t) = \bar{w}(t) + \int_0^t k(t-\tau)\bar{w}(\tau)d\tau, \ t \geq 0 \quad , \tag{3.15}$$

where k is the resolvent kernel associated with a', i.e. k is the unique solution of

$$k(t) + \int_0^t a'(t-\tau)k(\tau)d\tau = -a'(t), \ t \geq 0 \quad . \tag{3.16}$$

Using (3.15) to solve (3.1) for $\phi(u_x)_x$ in terms of u_{tt} yields

$$\phi(u_x(x,t))_x = u_{tt}(x,t) + \int_0^t k(t-\tau)u_{tt}(x,\tau)d\tau \qquad x \in B, \ t \geq 0 \quad . \tag{3.17}$$

After an integration by parts, this becomes

$$u_{tt}(x,t) + k(0)u_t(x,0) = \phi(u_x(x,t))_x + k(t)u_1(x) - \int_0^t k'(t-\tau)u_t(x,\tau)d\tau \quad , \ x \in B, \ t \geq 0 \quad , \tag{3.18}$$

where use has been made of (3.2). It follows from (3.16) that $k(0) = -a'(0)$, and thus the term $k(0)u_t$ has a damping effect if $a'(0) < 0$. This form of the equation is extremely convenient for many purposes.

Remark 3.4: If $\psi \equiv \phi$, then (2.21) also arises in a mathematical model for heat flow in materials with memory. For the heat flow problem, (3.3), (3.4), and (3.5) are still appropriate, but (3.6) should be replaced by $a(0) = 1$. This seemingly minor change leads to major differences in the analysis. The

memory term actually has a slightly stronger dissipative effect in this situation. (See [20], [6], and [30].)

For the general case with ψ different from ϕ, Dafermos and Nohel [7] exploited the positivity of $\chi'(0)$ and the strong positive definiteness of a to obtain global a priori estimates for solutions of initial-boundary value problems with $B = [0,1]$. They integrate by parts and use (2.14) and (2.22) to rewrite (2.21) in the form

$$u_{tt}(x,t) = \chi(u_x(x,t))_x + \int_0^t a(t-\tau)\psi(u_x(x,\tau))_{x\tau}d\tau \qquad (3.19)$$
$$+ a(t)\psi'(u_0'(x))u_0''(x) + g(x,t) \quad ,$$
$$x \in B,\ t > 0 \quad .$$

They obtain estimates for certain higher order derivatives directly from (3.19) and use the Poincaré inequality to estimate lower order derivatives. Their procedure yields global existence (and decay) of smooth solutions for small data with $B = [0,1]$ under boundary conditions (2.23), (2.24), or (2.25). However, due to the lack of Poincaré-type inequalities on all of space, their results do not apply to the pure initial value problem (2.21), (2.22) with $B = \mathbb{R}$.

Regarding ϕ, ψ, and a, they assume that

$$\phi,\ \psi \in C^3(\mathbb{R}),\ \phi(0) = \psi(0) = 0 \quad , \qquad (3.20)$$

$$\phi'(0) > 0,\ \psi'(0) > 0,\ \chi'(0) > 0 \quad , \qquad (3.21)$$

and that (3.4) and (3.5) hold. Their assumptions on the data and the conclusions of their existence theorems are essentially the same as those stated previously for initial-boundary value problems in the special case $\psi \equiv \phi$.

Subsequently, Hrusa and Nohel [17] established a global existence theorem for the Cauchy problem

$$u_{tt}(x,t) = \phi(u_x(x,t))_x + \int_0^t a'(t-\tau)\phi(u_x(x,\tau))_x d\tau \tag{3.22}$$
$$+ g(x,t),\ x \in \mathbb{R},\ t > 0\ ,$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \mathbb{R}\ . \tag{3.23}$$

We state a slightly simplified version of this result.

<u>Theorem 3.1</u>: Assume that (3.20), (3.21), (3.4), (3.5) hold, and that a satisfies some (mild) additional technical conditions. Then, there exists a constant $\mu > 0$ such that for each $u_0,\ u_1 : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ with

$$u_0 \in L^2_{\ell oc}(\mathbb{R}),\ u_0',\ u_0'',\ u_0''',\ u_1,\ u_1',\ u_1'' \in L^2(\mathbb{R})\ , \tag{3.24}$$

$$g,\ g_t,\ g_x \in C([0,\infty);\ L^2(\mathbb{R}))\ , \tag{3.25}$$

$$g,\ g_t \in L^1([0,\infty);\ L^2(\mathbb{R}))\ , \tag{3.26}$$

$$g_x,\ g_{tt} \in L^2([0,\infty);\ L^2(\mathbb{R}))\ , \tag{3.27}$$

and

$$\begin{aligned}(\int_{-\infty}^{\infty} \{u_0'(x)^2 + u_0''(x)^2 + u_0'''(x)^2 + u_1(x)^2 \\ + u_1'(x)^2 + u_1''(x)^2\}(x)dx)^{1/2} \\ + \sup_{t\geq 0} (\int_{-\infty}^{\infty} \{g^2 + g_t^2 + g_x^2\}(x,t)dx)^{1/2} \\ + \int_0^{\infty} (\int_{-\infty}^{\infty} \{g^2 + g_t^2\}(x,t)dx)^{1/2} dt \\ + (\int_0^{\infty} \int_{-\infty}^{\infty} \{g_x^2 + g_{tt}^2\}(x,t)dxdt)^{1/2} \\ < \mu\ ,\end{aligned} \tag{3.28}$$

the initial value problem (3.22), (3.23) has a unique solution $u \in C^2(\mathbb{R} \times [0,\infty))$ which satisfies (3.9). Moreover, as $t \to \infty$, (3.10) and (3.11) hold.

The proof combines certain estimates of Dafermos and Nohel [7] for higher order derivatives (which remain valid for $B = \mathbb{R}$) with a variant of MacCamy's procedure. (See [17] for the details.) The additional technical

assumptions on a (which are stated precisely in [17]) are not very restrictive; their purpose is to ensure integrability of certain resolvent kernels. In particular, relaxation functions of the form

$$a(t) := \sum_{j=1}^{N} \beta_j e^{-\alpha_j t} \quad , \quad t > 0 \quad , \tag{3.29}$$

with $\beta_j, \alpha_j > 0$ for $j = 1,2,\ldots,N$, which are commonly employed in applications of viscoelasticity theory, satisfy the assumptions of Theorem 3.1.

It is interesting to observe that if the relaxation function is a single decreasing exponential of the form $a(t) \equiv e^{-\alpha t}$, then (2.21) corresponds to a third order partial differential equation without memory. Indeed, in this case (2.21) becomes

$$\begin{aligned} u_{tt}(x,t) = \phi(u_x(x,t))_x &- \alpha \int_0^t e^{-\alpha(t-\tau)} \phi(u_x(x,\tau))_x d\tau \\ &+ g(x,t),\ x \in B,\ t > 0 \end{aligned} \tag{3.30}$$

and differentiation of (3.30) with respect to t yields

$$\begin{aligned} u_{ttt}(x,t) = \phi(u_x(x,t))_{xt} &- \alpha\phi(u_x(x,t))_x \\ &+ \alpha^2 \int_0^t e^{-\alpha(t-\tau)} \phi(u_x(x,\tau))_x d\tau \\ &+ g_t(x,t),\ x \in B,\ t > 0 \quad . \end{aligned} \tag{3.31}$$

It follows from (3.30) that

$$\begin{aligned} \alpha^2 \int_0^t e^{-\alpha(t-\tau)} \phi(u_x(x,\tau))_x d\tau &= \alpha\phi(u_x(x,t))_x \\ &+ \alpha g(x,t) - \alpha u_{tt}(x,t),\ x \in B,\ t > 0 \quad . \end{aligned} \tag{3.32}$$

Substituting (3.32) into (3.31) and using the definition of χ, we obtain

$$u_{ttt} + \alpha u_{tt} = \phi(u_x)_{xt} + \alpha\chi(u_x)_x + g_t + \alpha g \quad . \tag{3.33}$$

Greenberg [8] studied equation (3.33) with $B = [0,1]$ and $g \equiv 0$ under homogeneous Dirichlet boundary condtions. He derived a priori estimates which show that any sufficiently smooth and small solution decays to zero exponentially as $t \to \infty$. His analysis relies on the Poincaré inequality and

consequently does not apply if B is unbounded. In the next section, we prove Theorem 3.1 for equation (3.30).

In order to isolate the effects of nonlinearity in the memory term, Hrusa [16] has studied (2.21) in the special case that ϕ is linear (i.e., $\phi(\xi) \equiv c\xi$ for some constant $c > 0$), but ψ is allowed to be nonlinear. His results apply to initial-boundary value problems as well as pure initial value problems. It is shown in [16] that the local behavior of solutions of

$$u_{tt}(x,t) = cu_{xx}(x,t) + \int_0^t a'(t-\tau)\psi(u_x(x,\tau))_x d\tau + g(x,t), \quad x \in B, \ t > 0 , \tag{3.34}$$

is quite similar to that of solutions of the semilinear equation

$$u_{tt} = cu_{xx} + \psi(u_x) + g \quad . \tag{3.35}$$

In particular, a pointwise bound on u_x is sufficient to continue a C^2 solution u globally. Moreover, if ψ' is bounded, then (3.34) has globally defined smooth solutions, even for large initial data - independently of the sign of the memory term. (This requires only local assumptions on a.) Some decay results for solutions of (3.34) which allow the data to be large are also established in [16].

Several authors have analyzed the similar first order problem

$$u_t(x,t) + \phi(u(x,t))_x + \int_0^t a'(t-\tau)\phi(u(x,\tau))_x d\tau = 0, \quad x \in B, \ t > 0 , \tag{3.36}$$

$$u(x,0) = u_0(x) \quad , \quad x \in B \quad . \tag{3.37}$$

Equation (3.36) is simpler than (2.21) in that it is of first order, yet it retains many of the important qualitative features of (2.21). The chief motivation for studying (3.36) has been to gain insight into the behavior of solutions of (2.21).

If a' vanishes identically, then (3.36) reduces to the (scalar) conservation law

$$u_t + \phi(u)_x = 0 \quad . \tag{3.38}$$

It is well known that (3.38), (3.37) does not generally have a globally defined smooth solution, no matter how smooth u_0 is. Nohel [28] has shown that under reasonable conditions on ϕ, ψ, and a, the initial-boundary value problem (3.36), (3.37), with $B = [0,1]$ and periodic boundary conditions, has a unique global smooth solution if u_0 is sufficiently smooth and small. (Here, u_0 should be small in the $H^2(0,1)$ norm.) Malek-Madani and Nohel [24] have studied the formation of singularities in smooth solutions of (3.36) with $B = \mathbb{R}$. Under certain assumptions on ϕ, ψ, and a (which include (3.7) and convexity of ϕ), they give rather precise conditions on u_0 under which (3.36), (3.37) has a local smooth solution for which first derivatives become infinite in finite time.

Relatively little is known about weak solutions of (3.22) or (3.36). Dafermos and Hsiao [5] have established existence of global weak solutions (of class BV) to *systems of conservation* laws with memory in one space dimension for initial data having small total variation. They allow for very general types of memory terms in the equations. Their global results apply in several situations of physical interest (including the heat flow problem mentioned in Remark 3.4), but not to (3.22) under assumptions which are appropriate for viscoelastic solids of the Boltzmann type. (Their procedure does, however, yield local (in time) existence of BV solutions to (3.22) in this case.)

In [22], MacCamy studies several aspects of weak solutions of equation (3.36) with $\psi \equiv \phi$. He also discusses global existence of smooth solutions for small data, and the formation of singularities in smooth solutions. Greenberg and Hsiao [9] have studied the Riemann problem for a system which corresponds to (3.36) with $a(t) \equiv e^{-\alpha t}$, $\alpha > 0$. (See also [10].)

The results of Coleman and Gurtin [2] on wave propagation (which were discussed at the beginning of this section) hold for a more general class of materials with memory. For these materials, the displacement u obeys an equation of the form

$$u_{tt}(x,t) = \frac{\partial}{\partial x} G(u_x^t(x,\cdot)) + f(x,t), \; x \in B, \; t > 0 \quad , \tag{3.39}$$

where G is a smooth (nonlinear) functional defined on a function space of fading memory type, and for each $x \in B$, $t > 0$,

$$u_x^t(x,s) := u_x(x,t-s) \quad \forall\, s > 0 \quad , \tag{3.40}$$

i.e., u_x^t is the history up to time t of the strain. Under physically reasonable assumptions on G, Hrusa [14] has established global existence (and decay) of smooth solutions to certain history-boundary value problems for (3.39) with $B = [0,1]$ and suitably smooth and small data. See also [15] (and the references therein) for a more complete discussion of equation (3.39).

4. The Cauchy Problem with an Exponential Kernel

In this section we sketch the proof of global existence of smooth solutions to the initial value problem (3.22), (3.23) for sufficiently smooth and small data in the special case that the relaxation function is a decreasing exponential of the form $a(t) \equiv e^{-\alpha t}$. We also discuss the modifications required to treat more general relaxation functions. A linear rescaling of time shows that without loss of generality we may assume $\alpha = 1$. For simplicity we take $g \equiv 0$.

In particular, we consider the initial value problem

$$u_{tt}(x,t) = \phi(u_x(x,t))_x - \int_0^t e^{-(t-\tau)}\phi(u_x(x,\tau))_x d\tau \; , \quad x \in \mathbb{R}, \; t > 0 \quad , \tag{4.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \mathbb{R} \quad . \tag{4.2}$$

Observe that the corresponding equilibrium stress function is given by

$$\chi(\xi) := \phi(\xi) - \psi(\xi) \qquad \forall\ \xi \in \mathbb{R} \quad . \tag{4.3}$$

Concerning ϕ, ψ, and χ we make the assumptions

$$\phi,\ \psi \in C^3(\mathbb{R}),\ \phi(0) = \psi(0) = 0 \quad , \tag{4.4}$$

$$\phi'(0) > 0,\ \psi'(0) > 0,\ \chi'(0) > 0 \quad . \tag{4.5}$$

<u>Proposition</u>: Assume that (4.4) and (4.5) hold. Then, there exists a constant $\mu > 0$ such that for each $u_0, u_1 : \mathbb{R} \to \mathbb{R}$ with

$$u_0 \in L^2_{\ell oc}(\mathbb{R}),\ u_0', u_0'', u_0''', u_1, u_1',\ u_1'' \in L^2(\mathbb{R}) \tag{4.6}$$

and

$$\int_{-\infty}^{\infty} \{u_0'(x)^2 + u_0''(x)^2 + u_0'''(x)^2 + u_1(x)^2 + u_1'(x)^2 + u_1''(x)^2\} dx < \mu^2 \quad , \tag{4.7}$$

the initial value problem (4.1), (4.2) has a unique solution $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ with

$$u \in C^2(\mathbb{R} \times [0,\infty)) \quad , \tag{4.8}$$

$$u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in C([0,\infty);\ L^2(\mathbb{R})) \quad . \tag{4.9}$$

Moreover, as $t \to \infty$

$$u_{tt}, u_{tx}, u_{xx} \to 0 \quad \text{in} \quad L^2(\mathbb{R}) \quad , \tag{4.10}$$

$$u_t, u_x, u_{tt}, u_{tx}, u_{xx} \to 0 \quad \text{uniformly on} \quad \mathbb{R} \quad . \tag{4.11}$$

<u>Proof</u>: We choose a sufficiently small positive number δ and modify ϕ and ψ (and hence also χ) smoothly outside the interval $[-\delta,\delta]$ in such a way that ϕ' and ψ' are constant outside $[-2\delta,2\delta]$ and

$$\underline{\phi} < \phi'(\xi) < \overline{\phi},\ \underline{\psi} < \psi'(\xi) < \overline{\psi},\ \underline{\chi} < \chi'(\xi) < \overline{\chi} \qquad \forall\ \xi \in \mathbb{R} \quad , \tag{4.12}$$

where $\underline{\phi}$, $\overline{\phi}$, $\underline{\psi}$, $\overline{\psi}$, $\underline{\chi}$, $\overline{\chi}$ are positive constants satisfying

$$\bar{\chi}^2 - \phi \chi < 0 \quad . \tag{4.13}$$

(This can always be accomplished by virtue of (4.3) and (4.5).) There is no harm in making this modification because we will show a posteriori that $|u_x(x,t)| < \delta$ for all $x \in \mathbb{R}$, $t > 0$.

Making only minor changes in the proof of Theorem 2.1 of [7], one can establish the following local existence result: (4.1), (4.2) has a unique local solution u defined on a maximal time interval $[0,T_0)$, $T_0 > 0$, with

$$u \in C^2(\mathbb{R} \times [0,T_0)) \quad , \tag{4.14}$$

$$u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in C([0,T_0); L^2(\mathbb{R})) \quad . \tag{4.15}$$

Moreover, if

$$\sup_{t \in [0,T_0)} \int_{-\infty}^{\infty} \{u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2 + u_{xx}^2 + u_{ttt}^2 + u_{ttx}^2 + u_{txx}^2 + u_{xxx}^2\}(x,t)dx < \infty \quad , \tag{4.16}$$

then $T_0 = \infty$.

We now proceed to establish a priori estimates for the local solution u which will show that if (4.7) is satisfied with μ sufficiently small then (4.16) holds. For this purpose it is convenient to introduce

$$U_0 := \int_{-\infty}^{\infty} \{u_0'(x)^2 + u_0''(x)^2 + u_0'''(x)^2 + u_1(x)^2 + u_1'(x)^2 + u_1''(x)^2\}dx \quad , \tag{4.17}$$

$$\begin{aligned} E(t) := \max_{s \in [0,t]} & \int_{-\infty}^{\infty} \{u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2 + u_{xx}^2 \\ & + u_{ttt}^2 + u_{ttx}^2 + u_{txx}^2 + u_{xxx}^2\}(x,s)dx \\ & + \int_0^t \int_{-\infty}^{\infty} \{u_{tt}^2 + u_{tx}^2 + u_{xx}^2 + u_{ttt}^2 + u_{ttx}^2 \\ & + u_{txx}^2 + u_{xxx}^2\}(x,s)dxds \ , \quad t \in [0,T_0) \quad , \end{aligned} \tag{4.18}$$

$$\nu(t) := \sup_{\substack{x\in\mathbb{R} \\ s\in[0,t]}} \{u_x^2 + u_{tx}^2 + u_{xx}^2\}^{1/2}(x,s) \ , \ t \in [0,T_0) \ . \tag{4.19}$$

Throughout the remainder of this proof we use Γ to denote a (possibly large) generic positive constant which can be chosen independently of u_0, u_1, and T_0. The reader should note that all of the computations which follow are aimed at establishing an a priori bound of the form (4.43).

Differentiating (4.1) with respect to t and substituting for the integral term from (4.1) (as in the derivation of (3.33)) yields

$$u_{ttt} + u_{tt} = \phi(u_x)_{xt} + \chi(u_x)_x \ . \tag{4.20}$$

The required estimates will be obtained by combining several energy identities which we derive from (4.20).

We first multiply (4.20) by u_{tt} and integrate over space and time, performing several integrations by parts. The result of this calculation is

$$\begin{aligned} &\frac{1}{2}\int_{-\infty}^{\infty}\{u_{tt}^2 + \phi'(u_x)u_{tx}^2\}(x,t)dx + \int_0^t\int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds \\ &\quad + \int_{-\infty}^{\infty} \chi(u_x)u_{tx}(x,t)dx - \int_0^t\int_{-\infty}^{\infty} \chi'(u_x)u_{tx}^2(x,s)dxds \\ &\quad = \int_{-\infty}^{\infty}\{\tfrac{1}{2}u_{tt}^2 + \tfrac{1}{2}\phi'(u_x)u_{tx}^2 + \chi(u_x)u_{tx}\}(x,0)dx \\ &\quad\quad + \frac{1}{2}\int_0^t\int_{-\infty}^{\infty} \phi''(u_x)u_{tx}^3(x,s)dxds \\ &\quad\quad\quad \forall\, t \in [0,T_0) \ . \end{aligned} \tag{4.21}$$

Next, we multiply (4.20) by u_t and integrate as above, thereby obtaining

$$\begin{aligned} &\int_{-\infty}^{\infty}\{\tfrac{1}{2}u_t^2 + W(u_x)\}(x,t)dx + \int_0^t\int_{-\infty}^{\infty} \phi'(u_x)u_{tx}^2(x,s)dxds \\ &\quad + \int_{-\infty}^{\infty} u_t u_{tt}(x,t)dx - \int_0^t\int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds \\ &\quad = \int_{-\infty}^{\infty}\{\tfrac{1}{2}u_t^2 + W(u_x) + u_t u_{tt}\}(x,0)dx \\ &\quad\quad\quad \forall\, t \in [0,T_0) \ , \end{aligned} \tag{4.22}$$

where

$$W(\xi) := \int_0^{\xi} \chi(\eta)d\eta \qquad \forall\ \xi \in \mathbb{R} \ . \tag{4.23}$$

We multiply (4.22) by $(1-\varepsilon)$, with $0 < \varepsilon < 1$, and add the resulting equation to (4.21). After rearranging certain terms we have

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\{u_{tt}^2 + 2(1-\varepsilon)u_t u_{tt} + (1-\varepsilon)u_t^2\}(x,t)dx \\
&+ \int_{-\infty}^{\infty}\{\frac{1}{2}\phi'(u_x)u_{tx}^2 + \chi(u_x)u_{tx} + (1-\varepsilon)W(u_x)\}(x,t)dx \\
&+ \int_0^t \int_{-\infty}^{\infty}\{\varepsilon u_{tt}^2 + [\phi'(u_x) - \varepsilon\phi'(u_x)]u_{tx}^2\}(x,s)dxds \\
&= \int_{-\infty}^{\infty}\{\frac{1}{2}u_{tt}^2 + \frac{1}{2}\phi'(u_x)u_{tx}^2 + \chi(u_x)u_{tx} \\
&+ (1-\varepsilon)[W(u_x) + u_t u_{tt} + \frac{1}{2}u_t^2]\}(x,0)dx \\
&+ \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} \phi''(u_x)u_{tx}^3(x,s)dxds \qquad \forall\ t \in [0,T_0) \ .
\end{aligned} \tag{4.24}$$

We note that for each $\varepsilon \in (0,1)$ the first integrand on the left hand side of (4.24) is a positive definite quadratic form in u_t and u_{tt}. Moreover, we have

$$\varepsilon u_{tt}^2 + [\phi'(u_x) - \varepsilon\phi'(u_x)]u_{tx}^2 > \varepsilon u_{tt}^2 + (\underline{\phi} - \varepsilon\bar{\phi})u_{tx}^2 \tag{4.25}$$

which yields an obvious lower bound for the third integral on the left hand side of (4.24) if $\varepsilon < \underline{\phi}/\bar{\phi}$.

The second integral on the left hand side of (4.24) merits special attention. Observe that (4.4) and (4.12) imply

$$|\chi(\xi)| < \bar{\chi}|\xi|, \ W(\xi) > \frac{1}{2}\underline{\chi}\xi^2 \qquad \forall\ \xi \in \mathbb{R} \ . \tag{4.26}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{2}\phi'(u_x)u_{tx}^2 + \chi(u_x)u_{tx} + (1-\varepsilon)W(u_x) \\
&> \frac{1}{2}\underline{\phi}u_{tx}^2 - \bar{\chi}|u_x u_{tx}| + \frac{1}{2}(1-\varepsilon)\underline{\chi}u_x^2 \ .
\end{aligned} \tag{4.27}$$

A simple computation reveals that the right hand side of this last inequality is positive definite in u_x and u_{tx} for ε sufficiently small, by virture of (4.13).

Thus, by choosing ε small enough in (4.24), we conclude that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\{u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2\}(x,t)dx \\
&\qquad + \int_0^t \int_{-\infty}^{\infty}\{u_{tt}^2 + u_{tx}^2\}(x,s)dxds \\
&\quad \le \Gamma \int_{-\infty}^{\infty}\{u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2\}(x,0)dx \qquad (4.28)\\
&\qquad + \Gamma\left|\int_0^t \int_{-\infty}^{\infty} \phi''(u_x)u_{tx}^3(x,s)dxds\right| \quad \forall\, t \in [0,T_0) \ .
\end{aligned}$$

We observe that $u_{tt}(x,0) = \phi'(u_0'(x))u_0''(x)$ by (4.1) and (4.2), and since ϕ'' vanishes outside $[-2\delta,2\delta]$ we have

$$\begin{aligned}
&\left|\int_0^t \int_{-\infty}^{\infty} \phi''(u_x)u_{tx}^3(x,s)dxds\right| \\
&\quad \le \sup_{\substack{x\in\mathbb{R}\\ s\in[0,t]}} |\phi''(u_x)u_{tx}(x,s)| \int_0^t \int_{-\infty}^{\infty} u_{tx}^2(x,s)dxds \qquad (4.29)\\
&\quad \le \Gamma\nu(t)E(t) \quad \forall\, t \in [0,T_0) \ .
\end{aligned}$$

It now follows from (4.28) that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\{u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2\}(x,t)dx \\
&\qquad + \int_0^t \int_{-\infty}^{\infty}\{u_{tt}^2 + u_{tx}^2\}(x,s)dxds \qquad (4.30)\\
&\quad \le \Gamma U_0 + \Gamma\nu(t)E(t) \quad \forall\, t \in [0,T_0) \ .
\end{aligned}$$

To obtain our next identity we multiply (4.20) by u_{xx} and integrate as before, thus producing

$$\int_{-\infty}^{\infty}\{\tfrac{1}{2}\,\phi'(u_x)u_{xx}^2 - u_{xx}u_{tt}\}(x,t)dx$$

$$+ \int_0^t \int_{-\infty}^{\infty}\{\chi'(u_x)u_{xx}^2 - u_{xx}u_{tt}\}(x,s)dxds$$

$$- \frac{1}{2}\int_{-\infty}^{\infty} u_{tx}^2(x,t)dx \tag{4.31}$$

$$= \int_{-\infty}^{\infty}\{\tfrac{1}{2}\,\phi'(u_x)u_{xx}^2 - u_{xx}u_{tt} - \tfrac{1}{2}\,u_{tx}^2\}(x,0)dx$$

$$+ \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} \phi''(u_x)u_{tx}u_{xx}^2(x,s)dxds \quad \forall\ t \in [0,T_0) \ .$$

For each $\varepsilon > 0$ we have

$$|u_{xx}u_{tt}| \le \varepsilon u_{xx}^2 + \frac{1}{4\varepsilon}\,u_{tt}^2 \ . \tag{4.32}$$

We use (4.12) and (4.32) with ε sufficiently small to obtain lower bounds for the first two integrals on the left hand side of (4.31), and we majorize the right hand side as before. This yields the estimate

$$\int_{-\infty}^{\infty} u_{xx}^2(x,t)dx + \int_0^t \int_{-\infty}^{\infty} u_{xx}^2(x,s)dxds$$

$$- \Gamma \int_{-\infty}^{\infty}\{u_{tt}^2 + u_{tx}^2\}(x,t)dx - \Gamma \int_0^t \int_{-\infty}^{\infty} u_{tt}^2(x,s)dxds \tag{4.33}$$

$$\le \Gamma U_0 + \Gamma\nu(t)E(t) \quad \forall\ t \in [0,T_0) \ .$$

Combining (4.30) and (4.33) we conclude that

$$\int_{-\infty}^{\infty}\{u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2 + u_{xx}^2\}(x,t)dx$$

$$+ \int_0^t \int_{-\infty}^{\infty}\{u_{tt}^2 + u_{tx}^2 + u_{xx}^2\}(x,s)dxds \tag{4.34}$$

$$\le \Gamma U_0 + \Gamma\nu(t)E(t) \quad \forall\ t \in [0,T_0) \ .$$

We now must obtain similar bounds for third order derivatives of u. In order to avoid purely technical complications and highlight the main ideas, we give only formal derivations of the remaining energy identities (4.36), (4.37), and (4.39). The difficulty is that the local solution is not smooth enough to justify our formal procedure. However, all of these identities are in fact valid for our local

solution. They can be derived rigorously by approximation. (One way to do this is to use difference operators. See [7] for more details.)

Differentiation of (4.20) with respect to x yields

$$u_{xttt} + u_{xtt} = \phi(u_x)_{xxt} + \chi(u_x)_{xx} \quad . \tag{4.35}$$

We first multiply (4.35) by u_{xtt} and integrate as before. The outcome of this computation is

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\{u_{ttx}^2 + \phi'(u_x)u_{txx}^2\}(x,t)dx \\
&\quad + \int_0^t\int_{-\infty}^{\infty} u_{ttx}^2(x,s)dxds \\
&\quad + \int_{-\infty}^{\infty} \chi'(u_x)u_{xx}u_{txx}(x,t)dx \\
&\quad - \int_0^t\int_{-\infty}^{\infty} \chi'(u_x)u_{txx}^2(x,s)dxds \\
&= \int_{-\infty}^{\infty}\{\tfrac{1}{2}u_{ttx}^2 + \tfrac{1}{2}\phi'(u_x)u_{txx}^2 + \chi'(u_x)u_{xx}u_{txx}\}(x,0)dx \\
&\quad + \int_0^t\int_{-\infty}^{\infty}\{2\phi''(u_x)u_{xx}u_{txx}u_{ttx} + \phi''(u_x)u_{tx}u_{ttx}u_{xxx} \\
&\quad + \phi'''(u_x)u_{xx}^2u_{tx}u_{ttx} + \tfrac{1}{2}\phi''(u_x)u_{tx}u_{txx}^2 \\
&\quad + \chi''(u_x)u_{tx}u_{xx}u_{txx}\}(x,s)dxds \quad \forall\, t \in [0,T_0) \quad .
\end{aligned} \tag{4.36}$$

Next, we multiply (4.35) by u_{xt} and integrate as usual to obtain

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\{u_{tx}^2 + \chi'(u_x)u_{xx}^2\}(x,t)dx \\
&\quad + \int_0^t\int_{-\infty}^{\infty} \phi'(u_x)u_{txx}^2(x,s)dxds \\
&\quad + \int_{-\infty}^{\infty} u_{tx}u_{txx}(x,t)dx - \int_0^t\int_{-\infty}^{\infty} u_{ttx}^2(x,s)dxds \\
&= \int_{-\infty}^{\infty}\{\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}\chi'(u_x)u_{xx}^2 + u_{tx}u_{ttx}\}(x,0)dx \\
&\quad + \int_0^t\int_{-\infty}^{\infty}\{\phi''(u_x)u_{tx}u_{xx}u_{txx} \\
&\qquad + \tfrac{1}{2}\chi''(u_x)u_{tx}u_{xx}^2\}(x,s)dxds \quad \forall\, t \in [0,T_0) \quad .
\end{aligned} \tag{4.37}$$

Taking a suitable linear combination of (4.36) and (4.37) and estimating the left hand side from below and the right hand side from above as we did with (4.21) and (4.22) shows that

$$\begin{aligned} &\int_{-\infty}^{\infty}\{u_{tx}^2 + u_{xx}^2 + u_{ttx}^2 + u_{txx}^2\}(x,t)dx \\ &\quad + \int_0^t \int_{-\infty}^{\infty}\{u_{ttx}^2 + u_{txx}^2\}(x,s)dxds \\ &\quad\le \Gamma U_0 + \Gamma\{\nu(t) + \nu(t)^2\}E(t) \quad \forall\ t \in [0,T_0) \ . \end{aligned} \tag{4.38}$$

To obtain our final identity, we multiply (4.35) by u_{xxx} and integrate as usual. This yields

$$\begin{aligned} &\int_{-\infty}^{\infty}\{\tfrac{1}{2}\ \phi'(u_x)u_{xxx}^2 - u_{xxx}u_{ttx}\}(x,t)dx \\ &\quad + \int_0^t \int_{-\infty}^{\infty}\{\chi'(u_x)u_{xxx}^2 - u_{xxx}u_{ttx}\}(x,s)dxds \\ &\quad - \tfrac{1}{2}\int_{-\infty}^{\infty} u_{txx}^2(x,t)dx \\ &\quad = \int_{-\infty}^{\infty}\{\tfrac{1}{2}\ \phi'(u_x)u_{xx}^2 - u_{xxx}u_{txx} - \tfrac{1}{2}\ u_{txx}^2\}(x,0)dx \\ &\quad + \int_0^t \int_{-\infty}^{\infty}\{\tfrac{1}{2}\ \phi''(u_x)u_{tx}u_{xxx}^2 - 2\phi''(u_x)u_{xx}u_{txx}u_{xxx} \\ &\qquad - \phi''(u_x)u_{tx}u_{xxx}^2 - \phi'''(u_x)u_{tx}u_{xx}^2u_{xxx} \\ &\qquad - \chi''(u_x)u_{xx}^2u_{xxx}\}(x,s)dxds \quad \forall\ t \in [0,T_0) \ . \end{aligned} \tag{4.39}$$

Treating (4.39) in the same fashion as (4.31) and combining the result with (4.38) gives

$$\begin{aligned} &\int_{-\infty}^{\infty}\{u_{tx}^2 + u_{xx}^2 + u_{ttx}^2 + u_{txx}^2 + u_{xxx}^2\}(x,t)dx \\ &\quad + \int_0^t \int_{-\infty}^{\infty}\{u_{ttx}^2 + u_{txx}^2 + u_{xxx}^2\}(x,s)dxds \\ &\quad\le \Gamma U_0 + \Gamma\{\nu(t) + \nu(t)^2\}E(t) \quad \forall\ t \in [0,T_0) \ . \end{aligned} \tag{4.40}$$

Squaring (4.20) we get

$$u_{ttt}^2 < 4\phi'(u_x)u_{txx}^2 + 4\phi''(u_x)u_{tx}^2u_{xx}^2$$
$$+ 4u_{tt}^2 + 4\chi'(u_x)^2u_{xx}^2 \quad , \tag{4.41}$$

which, in conjunction with (4.34) and (4.40) yields

$$\int_{-\infty}^{\infty} u_{ttt}^2(x,t)dx + \int_0^t \int_{-\infty}^{\infty} u_{ttt}^2(x,s)dxds$$
$$< \Gamma U_0 + \Gamma\{\nu(t) + \nu(t)^2\}E(t) \quad \forall\ t \in [0,T_0) \quad . \tag{4.42}$$

Combining (4.34), (4.40), and (4.42) we finally deduce that

$$E(t) < \bar{\Gamma}U_0 + \bar{\Gamma}\{\nu(t) + \nu(t)^2\}E(t) \quad \forall\ t \in [0,T_0) \quad , \tag{4.43}$$

where $\bar{\Gamma}$ denotes a fixed positive constant which can be chosen independently of u_0, u_1, and T_0.

At this point, it should be noted that there are several other ways to obtain some of the estimates leading to (4.43). In particular, (4.1) can be used to express u_{xx} in terms of u_{tt} (and "small" correction terms) through an inverse linear Volterra operator. This eliminates the need for the identities (4.31) and (4.39).

We are now ready to synthesize the proof. We choose $\bar{E}$, $\mu > 0$ such that $\bar{\Gamma}\{(2\bar{E})^{1/2} + 2\bar{E}\} < \frac{1}{2}$, $\bar{E} < \delta^2$, and $\bar{\Gamma}\mu^2 < \frac{1}{4}\bar{E}$. Suppose now that (4.7) holds with the above choice of μ. It follows from the Sobolev embedding theorem that

$$\nu(t)^2 < 2E(t) \quad \forall\ t \in [0,T_0) \quad . \tag{4.44}$$

Therefore, we conclude from (4.43) that for any $t \in [0,T_0)$ with $E(t) < \bar{E}$ we actually have $E(t) < \frac{1}{2}\bar{E}$. Consequently, by continuity,

$$E(t) < \frac{1}{2}\bar{E} \quad \forall\ t \in [0,T_0) \quad , \tag{4.45}$$

provided that $E(0) < \frac{1}{2}\bar{E}$.

If necessary, we can always choose a smaller $\mu > 0$ such that (4.7) implies $E(0) < \frac{1}{2}\bar{E}$ and hence also that (4.45) is satisfied. This immediately yields $T_0 = \infty$ by virture of (4.16). It then follows from (4.18), (4.45), and standard embedding inequalities that (4.10) and (4.11) hold. Finally, we note that since $\bar{E} < \delta^2$, (4.19), (4.44), and (4.45) show that $|u_x(x,t)| < \delta$ for all $x \in \mathbb{R}$, $t > 0$. This completes the proof. ■

We close with a few remarks concerning the modifications required to treat the Cauchy problem with a more general relaxation function. As noted earlier, estimates for certain higher order derivatives can be obtained directly from (3.22) using the procedure of Dafermos and Nohel [7]. Under the assumptions of Theorem 3.1, equation (3.22) can be written in the form

$$
\begin{aligned}
u_{ttt} + a(0)^{-1}u_{tt} &= \phi(u_x)_{xt} + a(0)^{-1}\chi(u_x)_x \\
&\quad + a(0)^{-1}f + \frac{\partial}{\partial t}\,[K*(\phi(u_x)_x + f - u_{tt})] \quad ,
\end{aligned}
\tag{4.46}
$$

where the $*$ denotes convolution (with respect to the time variable) on $[0,t]$, i.e.

$$
(v*w)(t) := \int_0^t v(t-\tau)w(\tau)d\tau \quad , \quad t > 0 \quad , \tag{4.47}
$$

and K is the solution of a certain integral equation involving a' and a''.

Equation (4.46) is quite similar to (4.20) and this suggests a natural procedure to get estimates for the lower order derivatives. The chief difficulty lies in handling the convolution term. This is accomplished by rewriting it in several convenient equivalent forms involving derivatives on which we already have information. The details are carried out in [17].

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In this paper we survey recent results concerning global existence and decay of smooth solutions of certain quasilinear hyperbolic Volterra equations which provide models for the motion of one-dimensional viscoelastic solids of the Boltzmann type. We also sketch the derivation of these equations from physical principles, discuss the physically appropriate assumptions, and prove a special case of a new existence theorem for the Cauchy problem.

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